On Projectively Flat (α, β)-metrics

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Abstract

In this paper, we study a class of Finsler metrics defined by a Riemannian metric and a 1-form. We find a necessary and sufficient condition for the metric to be locally projectively flat in dimension $n \geq 3$.

1 Introduction

Projectively flat metrics on a convex open set in R^n are the solutions to the Hilbert's Fourth Problem. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. For Finsler metrics, the flag curvature is a natural extension of the sectional curvature. However the situation is much more complicated. It is well-known that every locally projectively flat Finsler metric is of scalar flag curvature, namely, the flag curvature is a scalar function on the tangent bundle, which might not necessarily be constant as in the Riemannian case. Thus locally projectively form a rich class of Finsler metrics. Below are two important examples defined by a Riemannian metric and a 1-form on the unit ball $B^n \subset R^n$: Let

$$\begin{aligned} \bar{\alpha} &= \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}, \\ \bar{\beta} &= \frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle}, \\ \lambda &= \frac{(1+\langle a, x\rangle)^2}{1-|x|^2}, \end{aligned}$$

where $a \in R^n$ is a constant vector with $|a| < 1$. Then

(a) $\bar{F} := \bar{\alpha} + \bar{\beta}$ is projectively flat on the unit ball $B^n(1) \subset R^n$ with constant flag curvature $\mathbf{K} = -1/4$ ([8]).

(b) $F := (\alpha + \beta)^2/\alpha$, where $\alpha = \lambda\bar{\alpha}$ and $\beta = \lambda\bar{\beta}$, is projectively flat on the unit ball $B^n(1) \subset R^n$ with zero flag curvature $\mathbf{K} = 0$ ([6]).

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These two examples inspire us to study projectively flat Finsler metrics $F = \alpha\phi(\beta/\alpha)$ defined by a Riemannian metric α and a 1-form β. Metrics in this form are called (α, β)-*metrics*. When $\phi = 1 + s$, we get Randers metrics $F = \alpha + \beta$. Randers metrics are the simplest (α, β)-metrics.

It is well-known that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed (see [3] [1]). For a general (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, if β is parallel with respect to α, then F is locally projectively flat if and only if α is locally projectively flat. This can be easily seen from (12) below.

The main purpose of this paper is to study and characterize locally projectively flat (α, β)-metrics which are not of Randers type.

We have the following

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $\mathcal{U}$ in the n-dimensional Euclidean space R^n $(n \geq 3)$, where $\phi(0) = 1$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i \neq 0$. Suppose that the following conditions: (a) β is not parallel with respect to α, (b) F is not in the form $F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ for some constants k and $\epsilon \neq 0$, and (c) $db \neq 0$ everywhere or $b = constant$ on $\mathcal{U}$. Then F is projectively flat on $\mathcal{U}$ if and only if the function $\phi = \phi(s)$ satisfies*

$$\Big\{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\Big\}\phi''(s) = (k_1 + k_2 s^2)\Big\{\phi(s) - s\phi'(s)\Big\}, \qquad (1)$$

$$b_{i|j} = 2\tau\Big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\Big\}, \qquad (2)$$

$$G^i_\alpha = \xi y^i - \tau\Big(k_1\alpha^2 + k_2\beta^2\Big)b^i, \qquad (3)$$

where $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$ and k_1, k_2 and k_3 are constants with $(k_2, k_3) \neq (0, 0)$.

When $(k_2, k_3) = (0, 0)$, the solution ϕ of (1) with $\phi(0) = 1$ is given by

$$\phi(s) = \sqrt{1 + k_1 s^2} + \epsilon s,$$

where ϵ is a constant. The (α, β)-metric defined by ϕ is of Randers type

$$F = \sqrt{\alpha^2 + k_1\beta^2} + \epsilon\beta.$$

For the above metric with $\epsilon \neq 0$, it is projectively flat if and only if β is closed and $\tilde{\alpha} := \sqrt{\alpha^2 + k_1\beta^2}$ is projectively flat, in other words, β is closed and α can be expressed as $\alpha = \sqrt{\tilde{\alpha}^2 - k_1\beta^2}$ where $\tilde{\alpha}$ is projectively flat. We do not consider this case in Theorem 1.1.

Consider the following functions:

$$\phi = e^s + \epsilon s, \qquad \phi = \frac{1}{1-s} + \epsilon s,$$

where ϵ is a constant. Clearly, they do not satisfy (1). Thus $F = \alpha \exp\Big(\frac{\beta}{\alpha}\Big) + \epsilon\beta$ (the expoential metric) and $F = \frac{\alpha^2}{\alpha - \beta} + \epsilon\beta$ (the Matsumoto metric) are projectively flat on $\mathcal{U}$ if and only if β is parallel with respect to α (Cf. [13], [5]). These metrics are projectively "hard". We conjecture that they never be of scalar flag curvature unless α is of constant sectional curvature and $\beta = 0$. On the other hand, there are many functions ϕ satisfying (1) for some constants k_i. Below are the most important ones.

$$\phi = 1 + s, \qquad \phi = 1 + \epsilon s + s^2, \tag{4}$$

$$\phi = 1 + \epsilon s + s \arctan(s), \qquad \phi = 1 + \epsilon s + 2s^2 - \frac{1}{3}s^4, \tag{5}$$

where ϵ is a constant. See [6] [12] for the metrics defined by $\phi = 1 + \epsilon s + s^2$, [14] for the metrics defined by $\phi = 1 + \epsilon s + s \arctan(s)$ and [7] for the metrics defined by $\phi = 1 + \epsilon s + 2s^2 - \frac{1}{3}s^4$.

Corollary 1.2 *If ϕ satisfies*

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s), \tag{6}$$

where $p \neq 0, r$ are constants, then it satisfies (1) with $k_1 = 1/p, k_2 = 0$ and $k_3 = (r-1)/p$. Then $F = \alpha\phi(\beta/\alpha)$ is projectively flat if and only if there is a scalar function $\tau = \tau(x)$ such that

$$b_{i|j} = \frac{2\tau}{p}\Big\{(p + b^2)a_{ij} + (r-1)b_i b_j\Big\}, \tag{7}$$

$$G^i_\alpha = \xi y^i - \frac{\tau}{p}\alpha^2 b^i. \tag{8}$$

This corollary slightly generalizes the theorem in [2] where the authors assume that $\phi = \phi(s)$ is analytic in s. The functions in (4) and (5) are particular solutions of (6). For these functions, one can find some special non-trivial solutions for (7) and (8). See [10] for some constructions of particular solutions. However, so far, we do not have any explicit examples satisfying (1)-(3) with $k_2 \neq 0$.

2 Preliminaries

Consider a Finsler metric $F = F(x,y)$ on an open domain $\mathcal{U} \subset R^n$. The geodesics of F satisfy the following equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where $G^i = G^i(x,y)$ are called the *geodesic coefficients* of F, which are given by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

F is said to be *projectively flat* in $\mathcal{U}$ if all geodesics are straight lines. This is equivalent to saying that the geodesic coefficients G^i of F take the following form

$$G^i = P(x,y)y^i. \tag{9}$$

There is another system of equations that characterizes projective flat geodesics. According to G. Hamel [4], F is projectively flat if and only if it satisfies

$$F_{x^m y^l} y^m - F_{x^l} = 0. \tag{10}$$

Equation (10) is more useful than (9) in the study of projectively flat (α,β)-metrics.

Let $\phi = \phi(s)$, $|s| < b_o$, be a positive C^∞ function satisfying the following

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b < b_o), \tag{11}$$

Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta = b_iy^i$ be a 1-form on a manifold M. Assuming that $\|\beta_x\|_\alpha < b_o$, the scalar function $F := \alpha\phi(s)$ where $s = \beta/\alpha$ is a Finsler metric which is called an (α,β)-metric. (α,β)-metrics form a special class of Finsler metrics. Most important, they are "computable" although the computation sometimes runs into very complicated situation.

Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ denote covariant derivative of β with respect to α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s_j := b^j s_{i|j}.$$

We can express the geodesic coefficients G^i of F in terms of the geodesic coefficients G^i_α of α and the covariant derivatives of β.

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{12}$$

where

$$\begin{aligned} P &= \alpha^{-1}\Theta\Big(- 2\alpha Q s_0 + r_{00}\Big) \\ Q^i &= \alpha Q {s^i}_0 + \Psi\Big(- 2\alpha Q s_0 + r_{00}\Big)b^i \end{aligned}$$

where

$$\begin{aligned} \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \\ Q &= \frac{\phi'}{\phi - s\phi'} \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

We have the following trivial lemmas.

Lemma 2.1 *If $\phi(0) = 1$ and $Q = k_1 s$, where k_1 is independent of s, then $\phi = \sqrt{1 + k_1 s^2}$.*

Lemma 2.2 *If $\phi(0) = 1$ and $2\Psi = \frac{k_1}{1+k_1 b^2}$, where k_1 is a number independent of s, then*

$$\phi = \sqrt{1 + k_1 s^2} + \epsilon s,$$

where ϵ is a number independent of s.

By (10), one can easily get the following

Lemma 2.3 ([12]) *An (α, β)-metric $F = \alpha\phi(s)$, where $s = \beta/\alpha$, is projectively flat on an open subset $\mathcal{U} \subset \mathrm{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(b_l \alpha - s y_l) = 0, \tag{13}$$

where $y_l := a_{lm} y^m$.

To simplify the equation (13), we shall prove the following

Theorem 2.4 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $U \subset R^n$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Suppose that (a) β is not parallel everywhere (b) F is not of Randers type at any point $x \in \mathcal{U}$; (c) either $db \neq 0$ everywhere or $b = constant \neq 0$ on U. Then F is projectively flat if and only if the function $\phi = \phi(s)$ satisfies*

$$\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''} = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \tag{14}$$

$$d\beta = 0 \tag{15}$$

$$r_{00} = 2\tau\Big\{\delta\beta^2 + \eta(b^2\alpha^2 - \beta^2)\Big\}, \tag{16}$$

$$G^i_\alpha = \xi y^i - \tau\Big(\lambda\beta^2 + \mu(b^2\alpha^2 - \beta^2)\Big)b^i, \tag{17}$$

where $\lambda, \mu, \delta, \eta$ and τ are scalar functions on U with $\delta = 0$ if $b = constant$.

3 The 1-form β is closed

In this section, we are going to prove the following

Lemma 3.1 *Suppose that Q/s is not independent of s. If an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is projectively flat on an open subset in $\mathcal{U}$ in R^n $(n > 2)$ and $b \neq 0$, then β is closed.*

Proof: Let $F = \alpha\phi(\beta/\alpha)$ be a projectively flat (α, β)-metric on $\mathcal{U}$, namely, its geodesics are straight lines. Fix an arbitrary point $x_o \in \mathcal{U} \subset R^n$. There is an affine transformation $\varphi = \mathbf{A}u + x_o : (u^i) \in R^n \to (x^i) \in R^n$ such that $\varphi(0) = x_o$ and $\alpha_{x_o} = \sqrt{a_{ij}v^iv^j}$ and $\beta_{x_o} = b_iv^i$ at $u = 0$ are given by

$$a_{ij} = \delta_{ij}, \qquad b_i = b\delta_{1i},$$

where $b := \|\beta_{x_o}\|_\alpha \neq 0$. The above identities hold only at $u = 0$. Since φ is affine, in the new coordinate system (u^i), the geodesics of $F = F(u, v)$ are still straight lines. Thus (13) holds for F with (u^i, v^i) in place of (x^i, y^i). At $u = 0$, we have

$$(\delta_{ml}\alpha^2 - v_mv_l)G^m_\alpha + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - sv_l) = 0, \qquad (18)$$

where $v_l := \delta_{lm}v^m$.

With x_o fixed, we make another change of coordinates: $(s, v^a) \to (v^i)$ by

$$v^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad v^a = v^a,$$

where

$$\bar{\alpha} := \sqrt{\sum_{a=2}^n (v^a)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

Let

$$\bar{r}_{10} := \sum_{a=2}^n r_{1a}v^a, \qquad \bar{r}_{00} := \sum_{a,b=2}^n r_{ab}v^av^b,$$

$$\bar{s}_{10} = \sum_{a=2}^n s_{1a}v^a, \qquad \bar{s}_0 := \sum_{a=2}^n s_av^a.$$

Note that

$$\bar{s}_0 = b\bar{s}_{10}, \qquad s_1 = bs_{11} = 0.$$

Plugging the above identities into (18) we get a system of equations in the following form

$$\Phi_l + \Psi_l\bar{\alpha} = 0,$$

where Φ_l and Ψ_l are polynomials in v^i. We must have

$$\Phi_l = 0, \qquad \Psi_l = 0.$$

Express

$$G^i_\alpha = \frac{1}{2}G^i_{jk}v^jv^k, \qquad G^i_{jk} = G^i_{kj}.$$

Let

$$\bar{G}^a_{10} = G^a_{1b} v^b, \quad \bar{G}^0_{10} = \bar{G}^0_{10} = G^a_{1b} v_a v^b, \quad \bar{G}^0_{00} = G^a_{bc} v_a v^b v^c.$$

where $v_a = \delta_{ab} v^b$.

For $l = 1$, by (18) we get

$$s\bar{G}^0_{00} = -s\bar{C}_0 \bar{\alpha}^2 + \Big\{ bQB\bar{s}_{10} + 2s\bar{A}_{10} \Big\} \bar{\alpha}^2 \tag{19}$$

$$s^2 A_{11} \bar{\alpha}^2 - 2s^2 \bar{G}^0_{10} + (b^2 - s^2)\bar{A}_{00} = 0. \tag{20}$$

For $l = a,\ 2 \le a \le n$, we get from (18) that

$$s\bar{G}^a_{00} = -sC_a \bar{\alpha}^2 + \Big\{ 2s\bar{A}_{10} + bQB\bar{s}_{10} \Big\} v^a, \tag{21}$$

$$\Big\{ 2sb^2 \bar{G}^a_{10} - s^3 A_{11} v^a + b^3 Q \bar{s}_{a0} \Big\} \bar{\alpha}^2 = s(b^2 - s^2) \Big\{ 2\bar{G}^0_{10} + \bar{A}_{00} \Big\} v^a. \tag{22}$$

Here

$$A_{ij} := G^1_{ij} + b\Psi r_{ij}, \quad B := 1 - 2\Psi b^2, \quad C_a = \frac{s}{b^2 - s^2} \Big\{ G^a_{11} s - bQ s_{1a} \Big\}.$$

$$\bar{C}_0 = C_a v^a, \quad \bar{A}_{10} = A_{1a} v^a, \quad \bar{A}_{00} = A_{ab} v^a v^b.$$

Note that contracting (21) with v_a yields (19) and contracting (22) with v_a yields (20). We can use (20) to eliminate A_{11} and A_{00} in (22).

$$\Big(2s\bar{G}^a_{10} + bQ\bar{s}_{a0} \Big) \bar{\alpha}^2 = 2s\bar{G}^0_{10} v^a. \tag{23}$$

Dividing (23) by $2s$ yields

$$\Big(\bar{G}^a_{10} + \frac{bQ}{2s} \bar{s}_{a0} \Big) \bar{\alpha}^2 = \bar{G}^0_{10} v^a. \tag{24}$$

Note that except for $bQ/(2s)$, other terms in (24) are independent of s . By assumption, Q/s is not independent of s. We conclude that $\bar{s}_{a0} = 0$, i.e.,

$$s_{ab} = 0. \tag{25}$$

In this case, (24) is reduced to

$$\bar{G}^a_{10} \bar{\alpha}^2 = \bar{G}^0_{10} v^a. \tag{26}$$

Differentiating (21) with respect to v^b and v^c, we get

$$2sG^a_{bc} = -2sC_a \delta_{bc} + \Big\{ (2sA_{1b} + bQBs_{1b})\delta^a_c + (2sA_{1c} + bQBs_{1c})\delta^a_b \Big\}. \tag{27}$$

Taking trace in (27) over $a = 2, \cdots, n$ yields

$$2sA_{1c} + bQBs_{1c} = \frac{2s}{n} \Big\{ sG^m_{mc} + C_c \Big\}. \tag{28}$$

Plugging (28) into (27), we get

$$G^a_{bc} - \frac{1}{n}\Big\{G^m_{mb}\delta^a_c + G^m_{mc}\delta^a_b\Big\} = C_a\delta_{bc} - \frac{1}{n}\Big\{C_b\delta^a_c + C_c\delta^a_b\Big\}. \tag{29}$$

By assumption, $n > 2$. For any $2 \le a \le n$, one can take $b = c \ne a$. In this case, (29) becomes

$$G^a_{bc} = C_a.$$

Note that $C_a = 0$ at $s = 0$. We get

$$G^a_{bc} = 0 \qquad (b = c \ne a).$$

Thus

$$C_a = 0, \qquad (|s| \le b).$$

By the definition of C_a, we get

$$G^a_{11} - \frac{bQ}{s} s_{1a} = 0.$$

By assumption, Q/s is not independent of s, we conclude that

$$s_{1a} = 0. \tag{30}$$

In this case, we also have

$$G^a_{11} = 0. \tag{31}$$

Since $s_{11} = 0$, it follows from (25) and (30) that

$$s_{ij} = 0. \tag{32}$$

4 Determining r_{ij} and G^i_α

In this section, we are going to derive two formulas for r_{ij} and G^i_α. We shall always assume that (a) F is projectively flat on $\mathcal{U}$, (b) F is not of Randers type at any point, (c) $b \ne 0$ at any point, (d) $dp \ne 0$ at any point or $b =$ *constant* and (e) β is not parallel everywhere. We continue to use the coordinate system (u^i, v^i) at $u = 0$. Express $\alpha = \sqrt{a_{ij}v^iv^j}$ and $\beta = b_iv^i$. We have at $u = 0$,

$$a_{ij} = \delta_{ij}, \qquad b_i = b\delta_{1i}.$$

In the previous section, we have shown that $C_a = 0$ and $s_{1b} = 0$ under the assumption that $n \ge 3$. Now (27) is reduced to

$$G^a_{bc} = A_{1b}\delta^a_c + A_{1c}\delta^a_b. \tag{33}$$

We can rewrite (33) as

$$G^a_{bc} - (G^1_{1b}\delta^a_c + G^1_{1c}\delta^a_b) = b\Psi(r_{1b}\delta^a_c + r_{1c}\delta^a_b). \tag{34}$$

Note that the left side is independent of s. If $r_{1c} \neq 0$ for some $2 \leq c \leq n$, then $b\Psi$ is independent of s. We can express Ψ as $2\Psi = \frac{k_1}{1+k_1 b^2}$ where k_1 is a number independent of s. By Lemma 2.2, ϕ is given by

$$\phi = \sqrt{1 + k_1 s^2} + \epsilon s,$$

where ϵ is a number independent of s. This case is excluded in the theorem. Thus we conclude that

$$r_{1b} = 0. \tag{35}$$

Then (34) is further reduced to the following

$$G^a_{bc} - (G^1_{1b}\delta^a_c + G^1_{1c}\delta^a_b) = 0. \tag{36}$$

It follows from (20) that

$$\begin{aligned} & s^2 \Big\{ G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) \Big\} + (b^2 - s^2) G^1_{ab} \\ & = -b\Psi \Big\{ s^2 r_{11}\delta_{ab} + (b^2 - s^2) r_{ab} \Big\}. \end{aligned} \tag{37}$$

Case I: $db \neq 0$ at $u = 0$. Observe that at $u = 0$,

$$[b^2]_{u^j} = 2b_i b_{i|j} = 2b_i r_{ij} + 2b_i s_{ij} = 2br_{1j} = 2br_{11}\delta_{1j}.$$

Thus $r_{11} \neq 0$. By (37), there are numbers $\lambda, \mu, \delta \neq 0$ and η independent of s such that

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}. \tag{38}$$

Actually, we may take

$$\delta = -br_{11}, \qquad \eta = -br_{22}, \lambda = \frac{1}{2}(G^1_{11} - 2G^2_{12}), \qquad \mu = \frac{1}{2}G^1_{22}.$$

Plugging (38) into (37) yields

$$\begin{aligned} & \delta \Big\{ G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) \Big\} = -\frac{b\lambda}{2} r_{11}\delta_{ab} \\ & \delta G^1_{ab} + \eta \Big\{ G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) \Big\} = -\frac{b\mu}{2} r_{11}\delta_{ab} - \frac{b\lambda}{2} r_{ab} \\ & \eta G^1_{ab} = -\frac{b\mu}{2} r_{ab}. \end{aligned}$$

Let τ be a number such that

$$r_{11} = 2b^2\delta\tau.$$

If $\mu\delta - \eta\lambda = 0$, then $2\Psi = \lambda/\delta$ is independent of s. We can express Ψ as $2\Psi = \frac{k_1}{1+k_1 b^2}$ where k_1 is a number independent of s. Then

$$\phi = \sqrt{1 + k_1 s^2} + \epsilon s,$$

where ϵ is a number independent of s. This is the case excluded in the theorem. Therefore we conclude that $\mu\delta - \eta\lambda \neq 0$. By this fact, we get from the above linear system that

$$r_{ab} = 2b^2\eta\tau\delta_{ab}, \tag{39}$$

$$G^1_{ab} = -b^3\mu\tau\delta_{ab}, \tag{40}$$

$$G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) = -b^3\lambda\tau\delta_{ab}, \tag{41}$$

Contracting (41) with v^a and v^b yields

$$\bar{G}^0_{10} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)\bar{\alpha}^2.$$

Plugging it into (26) gives

$$\bar{G}^a_{10} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)v^a.$$

Differentiating the above identity with respect to v^b, we get

$$G^a_{1b} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)\delta^a_b. \tag{42}$$

Finally, let us summarize what we have proved so far:

$$s_{11} = 0, \qquad s_{ab} = 0, \qquad s_{1a} = 0. \tag{43}$$

$$r_{11} = 2b^2\delta\tau, \qquad r_{ab} = 2b^2\eta\tau\delta_{ab}, \qquad r_{1a} = 0. \tag{44}$$

It is easy to see that (43) is equivalent to

$$s_{ij} = 0,$$

and (44) is equivalent to

$$r_{ij} = 2\tau\Big\{\delta b_i b_j + \eta(b^2\delta_{ij} - b_i b_j)\Big\}.$$

The above identities hold in (u^i) at $u = 0$. Back to the local system (x^i) at x_o, we get

$$r_{ij} = 2\tau\Big\{\delta b_i b_j + \eta(b^2 a_{ij} - b_i b_j)\Big\}. \tag{45}$$

By (31), (36), (40) and (42), we get

$$G^a_{11} = 0, \qquad G^1_{ab} = -b^3\mu\tau\delta_{ab},$$

$$G^1_{11} = k_1 - b^3\mu\tau, \qquad G^a_{1b} = \frac{1}{2}k_1\delta^a_b, \qquad G^1_{1a} = k_a, \qquad G^a_{bc} = k_b\delta^a_c + k_c\delta^a_b,$$

where k_i are numbers independent of s. It is easy to verify that the above identities are equivalent to

$$G^i_\alpha = \xi v^i - \tau\Big\{\lambda\beta^2 + \mu(b^2\alpha - \beta^2)\Big\}b^i, \tag{46}$$

where $\xi = k_j v^j$. The above identities hold in (u^i, v^i) at $u = 0$. Clearly, G^i_α take the same form in (x^i, y^i) at x_o (hence at any point x since x_o is chosen arbitrarily).

Case II: $b = \mathit{constant} \neq 0$. In this case, $r_{11} = 0$. We have proved that $s_{ij} = 0$ and $r_{1a} = 0$. Since we assume that β is not parallel, $(r_{ab}) \neq 0$. By (37), there are numbers λ, μ and $\eta \neq 0$ independent of s such that

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\eta(b^2 - s^2)}. \tag{47}$$

Plugging (47) into (37) yields

$$G^1_{ab} = -\frac{b\mu}{2\eta} r_{ab}, \tag{48}$$

$$G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) = -\frac{b\lambda}{2\eta} r_{ab}. \tag{49}$$

In this case, there is no restriction on r_{ab}.

Contracting (49) with v_a and v^b, we obtain that

$$\bar{G}^0_{10} = \frac{1}{2}\Big(G^1_{11}\bar{\alpha}^2 + \frac{b\lambda}{2\eta}\bar{r}_{00}\Big). \tag{50}$$

Plugging (50) into (26) yields

$$\Big(\bar{G}^a_{10} - \frac{1}{2}G^1_{11}v^a\Big)\bar{\alpha}^2 = \frac{b\lambda}{4\eta}\bar{r}_{00}v^a. \tag{51}$$

By (51), there is a number τ independent of s such that

$$r_{ab} = 2b^2\tau\eta\delta_{ab}, \tag{52}$$

and

$$G^a_{1b} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)\delta^a_b. \tag{53}$$

It follows from the fact $r_{11} = 0$, (35) and (52) that

$$r_{ij} = 2\tau\eta(b^2\delta_{ij} - b_i b_j). \tag{54}$$

Plugging (52) into (48) and (49) yields

$$G^1_{ab} = -2b^3\mu\tau\delta_{ab}, \tag{55}$$

$$G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) = -b^3\lambda\tau\delta_{ab}. \tag{56}$$

Contracting (56) with v^a and v^b yields

$$\bar{G}^0_{10} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)\bar{\alpha}^2.$$

Plugging it into (26) gives

$$\bar{G}^a_{10} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)v^a.$$

Differentiating the above identity with respect to v^b, we get

$$G^a_{1b} = \frac{1}{2}\Big(G^1_{11} + b^3\lambda\tau\Big)\delta^a_b. \tag{57}$$

It follows from (36) and (57) that there are numbers k_1 and k_a such that

$$G^1_{11} = k_1 - b^3\lambda\tau, \quad G^a_{1b} = \frac{1}{2}k_1\delta^a_b, \quad G^1_{1a} = k_a, \quad G^a_{bc} = k_b\delta^a_c + k_c\delta^a_b.$$

Together with (31) and (55) we get

$$G^i_\alpha = \xi v^i - \tau\Big(\lambda\beta^2 + \mu(b^2\alpha^2 - \beta^2)\Big)b^i, \tag{58}$$

where $\xi = k_i v^i$.

5 The equation on ϕ

To prove Theorem 1.1, we consider

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \tag{59}$$

where λ, μ, δ and μ are scalar functions with $(\lambda, \mu) \neq (0,0)$ and $(\delta, \eta) \neq (0,0)$, possibly depending on $b = \|\beta_x\|_\alpha$.

Lemma 5.1 *Assume that $\phi = \phi(s)$ with $\phi(0) = 1$ and $b \neq 0$ satisfies (59). Then $\phi^{(3)}(0) = \phi^{(5)}(0) = 0$ and one of the following holds:*

(i) if $\phi^{(4)}(0) + 3(\phi''(0))^2 \neq 0$,

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1 + k_2 s^2}{1 + k_1 b^2 + k_2 b^2 s^2 + k_3 s^2}, \tag{60}$$

where $k_1 = \phi''(0), k_2$ and k_3 are constants depending on $\phi''(0)$, $\phi^{(4)}(0)$ and $\phi^{(6)}(0)$.

(ii) if $\phi^{(4)}(0) + 3(\phi''(0))^2 = 0$, then

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1}{1 + k_1 b^2}, \tag{61}$$

where $k_1 = \phi''(0)$.

The equations (60) and (61) can be rewritten as one equation independent of b:

$$\Big\{1+k_1 s^2+k_2 s^4+k_3 s^2\Big\}\phi''(s)=(k_1+k_2 s^2)\Big\{\phi(s)-s\phi'(s)\Big\}. \qquad (62)$$

Proof: Rewrite (59) as follows

$$[\delta s^2+\eta(b^2-s^2)]\phi''=[\lambda s^2+\mu(b^2-s^2)][\phi-s\phi'+(b^2-s^2)\phi'']. \qquad (63)$$

Let

$$\phi=1+a_1 s+a_2 s^2+a_3 s^3+a_4 s^4+a_5 s^5+a_6 s^6+a_7 s^7+o(s^7).$$

Plugging the above Taylor expansion into (63), we get some linear equations on λ, μ, δ and η. We can actually solve these equations for λ, μ, δ and η based on the values of the following quantities

$$a_2, \qquad 1+2a_2 b^2, \qquad 2a_4+a_2^2.$$

Case 1: $a_2=0$ or $a_2=-\frac{1}{2b^2}$. Then by a comparison on the coefficients of the polynomials on both sides of (63), we conclude that $2a_4+a_2^2\neq 0$ and

$$\begin{aligned}\mu &= k_1\epsilon\\ \eta &= (1+k_1 b^2)\epsilon\\ \lambda &= (k_1+k_2 b^2)\epsilon\\ \delta &= (1+k_1 b^2+k_2 b^4+k_3 b^2)\epsilon.\end{aligned}$$

where ϵ is a number with $\epsilon\neq 0$ and k_i are given by

$$\begin{aligned}k_1 : &= 2a_2,\\ k_2 : &= 2\frac{a_4 a_2^2-5a_6 a_2+12a_4^2}{2a_4+a_2^2},\\ k_3 : &= -\frac{11a_4 a_2+5a_6+3a_2^3}{2a_4+a_2^2}.\end{aligned}$$

In this case,

$$\frac{\phi''(s)}{\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)}=\frac{k_1+k_2 s^2}{1+k_1 b^2+k_2 b^2 s^2+k_3 s^2}.$$

Case 2: $a_2\neq 0,\ -\frac{1}{2b^2}$, $1+2a_2 b^2\neq 0$ and $2a_4+a_2^2=0$. By a comparison on the coefficients of the polynomials on both sides of (63), we get $2a_6-a_2^3=0$ and

$$\begin{aligned}\mu &= k_1\epsilon\\ \eta &= (1+k_1 b^2)\epsilon\\ \lambda &= \frac{k_1}{1+k_1 b^2}\delta.\end{aligned}$$

where ϵ is a number with $\epsilon \neq 0$ and $k_1 = 2a_2$. In this case,

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1}{1 + k_1 b^2}.$$

Case 3: $a_2 \neq 0,\ -\frac{1}{2b^2}$, and $2a_4 + a_2^2 \neq 0$. By a comparison on the coefficients of the polynomials on both sides of (63), we still get

$$\begin{aligned}
\mu &= k_1\epsilon \\
\eta &= (1 + k_1 b^2)\epsilon \\
\lambda &= (k_1 + k_2 b^2)\epsilon \\
\delta &= (1 + k_1 b^2 + k_2 b^4 + k_3 b^2)\epsilon,
\end{aligned}$$

where ϵ is a number with $\epsilon \neq 0$ and k_i are given in Case 1. In this case

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1 + k_2 s^2}{1 + k_1 b^2 + k_2 b^2 s^2 + k_3 s^2}.$$

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